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Brownian and fractional Brownian stochastic currents via Malliavin calculus

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Abstract

By using Malliavin calculus and multiple Wiener–Itô integrals, we study the existence and the regularity of stochastic currents defined as Skorohod (divergence) integrals with respect to the Brownian motion and to the fractional Brownian motion. We consider also the multidimensional multiparameter case and we compare the regularity of the current as a distribution in negative Sobolev spaces with its regularity in the Watanabe spaces.

Keywords: Currents; Multiple stochastic integrals; Brownian motion; Fractional Brownian motion; Malliavin calculus

1. Introduction

The concept of current is proper of geometric measure theory. The simplest example is the functional

$$\varphi \mapsto \int_0^T \langle \varphi(\gamma(t)), \gamma'(t) \rangle_{\mathbb{R}^d} \, dt$$

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defined over the set of all smooth compact support vector fields $\varphi : \mathbb{R}^d \to \mathbb{R}^d$, with $\gamma : [0,T] \to \mathbb{R}^d$ being a rectifiable curve. This functional defines a vector valued distribution. Let us denote it by

$$\int_0^T \delta\big(x - \gamma(t)\big)\gamma'(t)\,dt. \tag{1}$$

We address for instance to the books [4,24,20,8], for definitions, results and applications.

The stochastic analog of 1-currents is a natural concept, where the deterministic curve $(\gamma(t))_{t\in[0,T]}$ is replaced by a stochastic process $(X_t)_{t\in[0,T]}$ in $\mathbb{R}^d$ and the stochastic integral must be properly interpreted. Several works deal with random currents, see for instance [17,10, 23,11,19,18,15,6,16,7]. Random currents have potential links with the area of stochastic analysis and differential geometry, see for instance [1] and some of the above mentioned works.

The difference between classical integration theory and random currents is the attempt to understand the latter as random distribution in the strong sense: random variables taking values in the space of distributions. The question is not simply how to define $\int_0^1 \langle \varphi(X_t), dX_t\rangle_{\mathbb{R}^d}$ for every given test functions, but when this operation defines, *for almost every realization* of the process X, a continuous functional on some space of test functions. This is related to (a particular aspect of) T. Lyons theory of rough paths: a random current of the previous form is a concept of pathwise stochastic integration. The degree of regularity of the random analog of expression (1) is a fundamental issue.

In this work we study the mapping defined as

$$\xi(x) = \int_{[0,T]^N} \delta(x - B_s)\,dB_s, \quad x \in \mathbb{R}^d,\ T > 0. \tag{2}$$

Here the integrator process is a d-dimensional Wiener process with multidimensional time parameter or a d-dimensional fractional Brownian motion.

A first direction of investigation is the regularity with respect to the variable $x \in \mathbb{R}^d$ of the mapping given by (2) in the (deterministic) Sobolev spaces $H^{-r}(\mathbb{R}^d;\mathbb{R}^d)$. This is the dual space to $H^r(\mathbb{R}^d,\mathbb{R}^d)$, or equivalently the space of all vector valued distributions φ such that $\int_{\mathbb{R}^d}(1 + |x|^2)^{-r}|\hat{\varphi}(x)|^2\,dx < \infty$, $\hat{\varphi}$ being the Fourier transform of φ. Some results on this have been recently obtained by [6], [5] or [7] using different techniques of the stochastic calculus. We propose here a new approach to this problem. The main tool is constituted by the Malliavin calculus based on the *Wiener–Itô chaos expansion*. Actually the delta Dirac function $\delta(x - B_s)$ can be understood as a distribution in the Watanabe sense (see [25]) and it has been the object of study by several authors, as [22], [12] or [13] in the Brownian motion case, or [2], [3] or [9] in the fractional Brownian motion case. In particular, it is possible to obtain the decomposition of the delta Dirac function into an orthogonal sum of multiple Wiener–Itô integrals and as a consequence it is easy to get the chaos expansion of the integral (2) since the divergence integral acts as a creation operator on Fock spaces (see Section 2). As it will be seen, the chaos expansion obtained will be useful to obtain the regularity properties of the mapping ξ. Another advantage of this method is the fact that it can be relatively easily extended to multidimensional settings.

Besides the estimation of the Sobolev regularity with respect to x, we are also interested to study the regularity with respect to ω, that is, as a functional in the Watanabe sense, of the integral (2). Our interest comes from the following observation. Consider $N = d = 1$. It is well known (see Nualart and Vives [22]) that for fixed $x \in \mathbb{R}$ we have

$$\delta(x - B_s) \in \mathbb{D}^{-\alpha,2} \quad \text{for any } \alpha > \frac{1}{2}$$

where $\mathbb{D}^{-\alpha,2}$ are the Watanabe (or Sobolev–Watanabe) spaces introduced in Section 2. On the other side, more or less surprisedly the same order of regularity holds with respect to x; for fixed ω, the mapping $g(x) = \delta(x - B_s(\omega))$ belongs to the negative Sobolev space $H^{-r}(\mathbb{R};\mathbb{R})$ for every $r > \frac{1}{2}$. Indeed, since the Fourier transform of g is $\hat{g}(x) = e^{-ixB_s}$ we have

$$|g|^2_{H^{-r}(\mathbb{R};\mathbb{R})} = \int_{\mathbb{R}} \left|\hat{g}(x)\right|^2 \left(1 + x^2\right)^{-r} dx$$

and this is finite if and only if $r > \frac{1}{2}$.

One can ask the question if this similarity of the order of regularity in the deterministic and stochastic Sobolev space still holds for the functional ξ defined by 2; and actually in the case of dimension $d = 1$ the above property still holds: we have the same regularity of ξ both with respect to x and with respect to ω.

One can moreover ask if it holds for other Gaussian processes. The answer to this question is negative, since we show that (actually, this has also been proved in [7] but using another integral with respect to fBm) in the fractional Brownian motion case the mapping (2) belongs to $H^{-r}(\mathbb{R};\mathbb{R})$ for any $r > \frac{1}{2H} - \frac{1}{2}$ (as a function in x) and to $\mathbb{D}^{-\alpha,2}$ with $\alpha > \frac{3}{2} - \frac{1}{2H}$ (as a function on ω).

We organized our paper as follows. Section 2 contains some preliminaries on Malliavin calculus and multiple Wiener–Itô integrals. Section 3 contains a discussion about random distribution where we unify the definition of the quantity $\delta(x - X_s(\omega))$ (X is a Gaussian process on $\mathbb{R}^d$) which in principle can be understood as a distribution with respect to x and also as a distribution in the Watanabe sense when it is regarded as o function of ω. In Section 4 we study the existence and the regularity of the stochastic currents driven by a N parameter Brownian motion in $\mathbb{R}^d$ while in Section 5 concerning the same problem when the driving process is the fractional Brownian motion. In Section 6 we give the regularity of the integral (2) with respect to ω and we compare it with its regularity in x.

2. Preliminaries

Here we describe the elements from stochastic analysis that we will need in the paper. Consider $\mathcal{H}$ a real separable Hilbert space and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathcal{A}, P)$, that is a centered Gaussian family of random variables such that $\mathbf{E}(B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$. Denote by I_n the multiple stochastic integral with respect to B (see [21]). This I_n is actually an isometry between the Hilbert space $\mathcal{H}^{\odot n}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{n!}}\| \cdot \|_{\mathcal{H}^{\otimes n}}$ and the Wiener chaos of order n which is defined as the closed linear span of the random variables $H_n(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_n is the Hermite polynomial of degree $n \geqslant 1$

$$H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right), \quad x \in \mathbb{R}.$$

The isometry of multiple integrals can be written as: for m, n positive integers,

$$\mathbf{E}\big(I_n(f)I_m(g)\big) = n!\langle f, g\rangle_{\mathcal{H}^{\otimes n}} \quad \text{if } m = n,$$
$$\mathbf{E}\big(I_n(f)I_m(g)\big) = 0 \quad \text{if } m \neq n. \tag{3}$$

It also holds that

$$I_n(f) = I_n(\tilde{f})$$

where $\tilde{f}$ denotes the symmetrization of f defined by $\tilde{f}(x_1, \ldots, x_x) = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

We recall that any square integrable random variable which is measurable with respect to the σ-algebra generated by B can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n\geqslant 0} I_n(f_n) \tag{4}$$

where $f_n \in \mathcal{H}^{\odot n}$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}[F]$.

Let L be the Ornstein–Uhlenbeck operator

$$LF = -\sum_{n\geqslant 0} nI_n(f_n)$$

if F is given by (4).

For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev–Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \big\|(I - L)^{\alpha/2}\big\|_{L^p(\Omega)}$$

where I represents the identity. In this way, a random variable F as in (4) belongs $\mathbb{D}^{\alpha,2}$ if and only if

$$\sum_{n\geqslant 0}(1+n)^{\alpha}\big\|I_n(f_n)\big\|^2_{L^2(\Omega)} = \sum_{n\geqslant 0}(1+n)^{\alpha} n!\|f_n\|^2_{\mathcal{H}^{\otimes n}} < \infty.$$

We denote by D the Malliavin derivative operator that acts on smooth functions of the form $F = g(B(\varphi_1), \ldots, B(\varphi_n))$ (g is a smooth function with compact support and $\varphi_i \in \mathcal{H}$)

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\big(B(\varphi_1), \ldots, B(\varphi_n)\big)\varphi_i.$$

The operator D is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}(\mathcal{H})$. The adjoint of D is denoted by δ and is called the divergence (or Skorohod) integral. It is a continuous operator from $\mathbb{D}^{\alpha,p}(\mathcal{H})$ into $\mathbb{D}^{\alpha-1,p}$. For adapted integrands, the divergence integral coincides to the classical Itô integral. We will use the notation

$$\delta(u) = \int_0^T u_s \, dB_s.$$

Let u be a stochastic process having the chaotic decomposition $u_s = \sum_{n\geqslant 0} I_n(f_n(\cdot, s))$ where $f_n(\cdot, s) \in \mathcal{H}^{\otimes n}$ for every s. One can prove that $u \in \operatorname{Dom}\delta$ if and only if $\tilde{f}_n \in \mathcal{H}^{\otimes(n+1)}$ for every $n \geqslant 0$, and $\sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$ converges in $L^2(\Omega)$. In this case,

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n) \quad \text{and} \quad \mathbf{E}\left|\delta(u)\right|^2 = \sum_{n=0}^{\infty} (n+1)! \|\tilde{f}_n\|^2_{\mathcal{H}^{\otimes(n+1)}}.$$

In the present work we will consider divergence integral with respect to a Brownian motion in $\mathbb{R}^d$ as well as with respect to a fractional Brownian motion in $\mathbb{R}^d$.

Throughout this paper we will denote by $p_s(x)$ the Gaussian kernel of variance $s > 0$ given by $p_s(x) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}}$, $x \in \mathbb{R}$ and for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ by $p_s^d(x) = \prod_{i=1}^d p_s(x_i)$.

3. Random distributions

We study now the regularity of the stochastic integral given by (2). Our method is based on the Wiener–Itô chaos decomposition. Let X be isonormal Gaussian process with variance $R(s,t)$. We will use the following decomposition of the delta Dirac function (see Nualart and Vives [22], Imkeller et al. [12], Kuo [14], Eddahbi et al. [3]) into orthogonal multiple Wiener–Itô integrals

$$\delta(x - X_s) = \sum_{n\geqslant 0} R(s)^{-\frac{n}{2}} p_{R(s)}(x) H_n\left(\frac{x}{R(s)^{1/2}}\right) I_n\left(1_{[0,s]}^{\otimes n}\right) \tag{5}$$

where $R(s) := R(s,s)$, $p_{R(s)}$ is the Gaussian kernel of variance $R(s)$, H_n is the Hermite polynomial of degree n and I_n represents the multiple Wiener–Itô integral of degree n with respect to the Gaussian process X as defined in the previous section.

A key element of the entire work is the quantity $\delta(x - B_s)$. This can be understood as a generalized random variable in some Sobolev–Watanbe space as well as a generalized function with respect to the variable x. The purpose of this section is to give an unitary definition of the delta Dirac as a random distribution.

Let $D(\mathbb{R}^d)$ be the space of smooth compact support functions on $\mathbb{R}^d$ and let $D'(\mathbb{R}^d)$ be its dual, the space of distributions, endowed with the usual topologies. We denote by $\langle S, \varphi\rangle$ the dual pairing between $S \in D'(\mathbb{R}^d)$ and $\varphi \in D(\mathbb{R}^d)$. We say that a distribution $S \in D'(\mathbb{R}^d)$ is of class $L^p_{loc}(\mathbb{R}^d)$, $p \geqslant 1$, if there is $f \in L^1_{loc}(\mathbb{R}^d)$ such that $\langle S, \varphi\rangle = \int_{\mathbb{R}^d} f(x)\varphi(x)\,dx$ for every $\varphi \in D(\mathbb{R}^d)$. In this case we also say that the distribution is given by a function.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, with expectation denoted by $\mathbf{E}$.

Definition 1. We call random distribution (on $\mathbb{R}^d$, based on $(\Omega, \mathcal{A}, P)$) a measurable mapping $\omega \mapsto S(\omega)$ from $(\Omega, \mathcal{A})$ to $D'(\mathbb{R}^d)$ with the Borel σ-algebra.

Given a random distribution S, for every $\varphi \in D(\mathbb{R}^d)$ the real valued function $\omega \mapsto \langle S(\omega), \varphi\rangle$ is measurable. The converse is also true.

Similarly to the deterministic case, we could say that a random distribution S is of class $L^0(\Omega, L^1_{loc}(\mathbb{R}^d))$ if there exists a measurable function $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $f(\omega, \cdot) \in L^1_{loc}(\mathbb{R}^d)$ for P-a.e. $\omega \in \Omega$, such that $\langle S(\omega), \varphi\rangle = \int_{\mathbb{R}^d} f(\omega, x)\varphi(x)\,dx$ for every $\varphi \in D(\mathbb{R}^d)$. In this case we also say that the distribution is given by a random field. However, this concept is restrictive if we have to deal with true distributions.

There is an intermediate concept, made possible by the simultaneous presence of the two variables $\omega \in \Omega$ and $x \in \mathbb{R}^d$. One could have that the random distribution is given by a function with respect to the x variable, but at the price that it is distribution-valued (in the Sobolev–Watanabe sense) in the ω variable. To this purpose, assume that a real separable Hilbert space $\mathcal{H}$ is given, an isonormal Gaussian process $(W(h), h \in \mathcal{H})$ on $(\Omega, \mathcal{A}, P)$ is given. For $p > 1$ and $\alpha \in \mathbb{R}$, denote by $\mathbb{D}^{\alpha,p}$ the Sobolev–Watanabe space of generalized random variables on $(\Omega, \mathcal{A}, P)$, defined in terms of Wiener chaos expansion.

Definition 2. Given $p \geqslant 1$ and $\alpha \leqslant 0$, a random distribution S is of class $L^1_{loc}(\mathbb{R}^d; \mathbb{D}^{\alpha,p})$ if $\mathbf{E}[|\langle S, \varphi\rangle|^p] < \infty$ for all $\varphi \in D(\mathbb{R}^d)$ and there exists a function $f \in L^1_{loc}(\mathbb{R}^d; \mathbb{D}^{\alpha,p})$ such that

$$\langle S, \varphi\rangle = \int_{\mathbb{R}^d} \varphi(x) f(x)\,dx$$

for every $\varphi \in D(\mathbb{R}^d)$.

Let us explain the definition. The integral $\int_{\mathbb{R}^d} \varphi(x) f(x)\,dx$ is of Bochner type (the integral of the $\mathbb{D}^{\alpha,p}$-valued function $x \mapsto \varphi(x) f(x)$). The integral $\int_{\mathbb{R}^d} \varphi(x) f(x)\,dx$ is thus, a priori, an element of $\mathbb{D}^{\alpha,p}$. The random variable $\omega \mapsto \langle S(\omega), \varphi\rangle$, due to the assumption $\mathbf{E}[|\langle S, \varphi\rangle|^p] < \infty$, is an element of $\mathbb{D}^{\alpha',p}$ for all $\alpha' \leqslant 0$. The definition requires that they coincide, as elements of $\mathbb{D}^{\alpha,p}$, a priori. A fortiori, since $\omega \mapsto \langle S(\omega), \varphi\rangle$ is an element of $\mathbb{D}^{0,p}$, the same must be true for $\int_{\mathbb{R}^d} \varphi(x) f(x)\,dx$. Thus, among the consequences of the definition there is the fact that $\int_{\mathbb{R}^d} \varphi(x) f(x)\,dx \in \mathbb{D}^{0,p}$ although $f(x)$ lives only in $\mathbb{D}^{\alpha,p}$.

The following theorem treats our main example. Given any d-dimensional random variable X on $(\Omega, \mathcal{A}, P)$, let S be the random distribution defined as

$$\bigl\langle S(\omega), \varphi\bigr\rangle := \varphi\bigl(X(\omega)\bigr), \quad \varphi \in D\bigl(\mathbb{R}^d\bigr).$$

We denote this random distribution by

$$\delta\bigl(x - X(\omega)\bigr)$$

and we call it the delta Dirac at $X(\omega)$. Denote Hermite polynomials by $H_n(x)$, multiple Wiener integrals by I_n, Gaussian kernel of variance σ^2 by $p_{\sigma^2}(x)$.

If $\mathcal{H}_1, \ldots, \mathcal{H}_d$ are real and separable Hilbert spaces, a d-dimensional isonormal process is defined on the product space $(\Omega, \mathcal{A}, P)$, $\Omega = \Omega_1 \times \cdots \times \Omega_d$, $\mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_d$, $P = P_1 \otimes \cdots \otimes P_d$ as a vector with independent components $((W^1(h_1), \ldots, W^d(\varphi_d))$, $h_1 \in \mathcal{H}_1, \ldots, h_d \in \mathcal{H}_d)$ where for every $i = 1, \ldots, d$, $(W^i(h_i), h_i \in \mathcal{H}_i)$ is an one-dimensional isonormal process on $(\Omega_i, \mathcal{A}_i, P_i)$. We denote by $\mathbf{E}_i$ the expectation on $(\Omega_i, \mathcal{A}_i, P_i)$ and by $\mathbf{E}$ the expectation on $(\Omega, \mathcal{A}, P)$.

Theorem 1. *Let $\mathcal{H}_i$, $i = 1, \ldots, d$, be real separable Hilbert spaces. Consider $((W^1(h_1), \ldots, W^d(\varphi_d))$, $h_1 \in \mathcal{H}_1, \ldots, h_d \in \mathcal{H}_d)$ a d-dimensional isonormal Gaussian process on $(\Omega, \mathcal{A}, P)$ as above. Then, for every $h_i \in \mathcal{H}_i$, $i = 1, \ldots, d$, the random distribution $\delta(\cdot - W(h))$ is of class $L^1_{loc}(\mathbb{R}^d; \mathbb{D}^{\alpha,2})$ for some $\alpha < 0$ and the associated element f of $L^1_{loc}(\mathbb{R}^d; \mathbb{D}^{\alpha,2})$ is*

$$f(x) = \sum_{n_1,\ldots,n_d \geqslant 0} \prod_{i=1}^{d} |h_i|^{-n_i} p_{|h_i|^2}(x_i) H_{n_i}\left(\frac{x_i}{|h_i|}\right) I^i_{n_i}\big(h_i^{\otimes n_i}\big)$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $I^i_{n_i}$, $i = 1, \ldots, d$, denotes the multiple Wiener–Itô integral with respect to the Wiener process $W^{(i)}$.

We denote this $\mathbb{D}^{\alpha,2}$-valued function $f(x)$ by $\delta(x - W(h))$.

Proof. Let us consider first the case $d = 1$. In this case

$$f(x) = \sum_{n \geqslant 0} |h|^{-n} p_{|h|^2}(x) H_n\left(\frac{x}{|h|}\right) I_n\big(h^{\otimes n}\big).$$

We have to prove that

$$\varphi\big(W(h)\big) = \int_{\mathbb{R}} \varphi(x) f(x)\, dx$$

for every test function φ. We have

$$\begin{aligned}
\int_{\mathbb{R}} \varphi(x) f(x)\, dx &= \sum_{n \geqslant 0} |h|^{-n} I_n\big(h^{\otimes n}\big) \int_{\mathbb{R}} \varphi(x) p_1\left(\frac{x}{|h|}\right) H_n\left(\frac{x}{|h|}\right) dx \\
&= \sum_{n \geqslant 0} |h|^{-n+1} I_n\big(h^{\otimes n}\big) \int_{\mathbb{R}} \varphi\big(|h|x\big) H_n(x) p_1(x)\, dx.
\end{aligned}$$

On the other hand, by Stroock's formula we can write

$$\varphi\big(W(h)\big) = \sum_{n \geqslant 0} \frac{1}{n!} I_n\big(D^{(n)} \varphi\big(W(h)\big)\big) = \sum_{n \geqslant 0} \frac{1}{n!} I_n\big(h^{\otimes n}\big) \mathbf{E}\big(\varphi^{(n)}\big(W(h)\big)\big)$$

and

$$\begin{aligned}
\mathbf{E}\big(\varphi^{(n)}\big(W(h)\big)\big) &= \int_{\mathbb{R}} \varphi^{(n)}(x) p_{|h|^2}(x)\, dx = |h| \int_{\mathbb{R}} \varphi^{(n)}\big(x|h|\big) p_1(x)\, dx \\
&= (-1) \int_{\mathbb{R}} \varphi^{(n-1)}\big(x|h|\big) \big(p_1(x)\big)'\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \cdots \\
&= (-1)^n |h|^{n-1} \int_{\mathbb{R}} \varphi\big(x|h|\big)\big(p_1(x)\big)^{(n)}\, dx.
\end{aligned}
$$

By using the recurrence formula for the Gaussian kernel p_1

$$
\big(p_1(x)\big)^{(n)} = (-1)^n n! p_1(x) H_n(x)
$$

we get

$$
\varphi\big(W(h)\big) = \sum_{n\geqslant 0} \int_{\mathbb{R}} \varphi\big(x|h|\big) H_n(x) p_1(x)\, dx = \int_{\mathbb{R}} \varphi(x) f(x)\, dx.
$$

Concerning the case $d \geqslant 2$, note that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \varphi(x) f(x)\, dx \\
&\quad = \sum_{n_1,\ldots,n_d\geqslant 0} \prod_{i=1}^{d} \left(|h_i|^{-n_i} I_{n_i}^i\big(h_i^{\otimes n_i}\big) \int_{\mathbb{R}^d} dx\, \varphi(x) p_{|h_i|^2}(x_i) H_{n_i}\left(\frac{x_i}{|h_i|}\right)\right) \\
&\quad = \sum_{n_1,\ldots,n_d\geqslant 0} \prod_{i=1}^{d} \left(|h_i|^{-n_i+1} I_{n_i}^i\big(h_i^{\otimes n_i}\big) \int_{\mathbb{R}^d} \varphi\big(|h_1|x_1,\ldots,|h_d|x_d\big) p_1(x_i) H_{n_i}(x_i)\, dx\right).
\end{aligned}
$$

Let us apply Stroock's formula in several steps. First we apply Stroock's formula with respect to the component W^i

$$
\begin{aligned}
\varphi\big(W(h)\big) = \varphi\big(W^1(h_1),\ldots,W^d(h_d)\big) &= \sum_{n_1\geqslant 0} \frac{1}{n_1!} I_{n_1}^1\big(\mathbf{E}_1\big[D^{(n),1}\varphi\big(W^1(h_1),\ldots,W^d(h_d)\big)\big]\big) \\
&= \sum_{n_1\geqslant 0} \frac{1}{n_1!} I_{n_1}^1\big(h_1^{\otimes n_1}\big)\mathbf{E}_1\left(\frac{\partial^{n_1}\varphi}{\partial x_1^{n_1}}\big(W^1(h_1),\ldots,W^d(h_d)\big)\right)
\end{aligned}
$$

and then with respect to W^2

$$
\frac{\partial^{n_1}\varphi}{\partial x_1^{n_1}}\big(W^1(h_1),\ldots,W^d(h_d)\big) = \sum_{n_2\geqslant 0} \frac{1}{n_2!} I_{n_2}^2\big(h_2^{\otimes n_2}\big)\mathbf{E}_2\left(\frac{\partial^{n_1+n_2}\varphi}{\partial x_1^{n_1} x_2^{n_2}}\right)\big(W^1(h_1),\ldots,W^d(h_d)\big).
$$

We will obtain

$$\varphi\big(W(h)\big) = \sum_{n_1,\dots,n_d \geqslant 0} \prod_{i=1}^{d} \frac{1}{n_i!} I_{n_i}^i\big(h_i^{\otimes n_i}\big) \mathbf{E}\bigg(\frac{\partial^{n_1+\dots+n_d}\varphi}{\partial x_1^{n_1}\dots x_d^{n_d}}\big(W^1(h_1),\dots,W^d(h_d)\big)\bigg)$$
$$= \sum_{n_1,\dots,n_d \geqslant 0} \prod_{i=1}^{d} \frac{1}{n_i!} I_{n_i}^i\big(h_i^{\otimes n_i}\big) \int_{\mathbb{R}^d} \frac{\partial^{n_1+\dots+n_d}\varphi}{\partial x_1^{n_1}\dots x_d^{n_d}}(x) p_{|h_i|^2}(x_i)\,dx$$

and now it suffices to follow the case $d = 1$ by integrating by parts and using the recurrence formula for the Gaussian kernel. □

4. Regularity of Brownian currents with respect to x

Let $(B_t)_{t\in[0,T]^N}$ be a Gaussian isonormal process. For every $n \geqslant 1$, $x \in \mathbb{R}$ and $s = (s_1,\dots,s_N) \in [0,T]^N$ we will use the notation

$$a_n^x(s) = R(s)^{-\frac{n}{2}} p_{R(s)}(x) H_n\bigg(\frac{x}{R(s)^{1/2}}\bigg) \tag{6}$$

with $R(s) = \mathbf{E}B_s^2 = s_1\dots s_N$. We will start by the following very useful calculation which plays an important role in the sequel.

Lemma 1. *Let a_n^x be given by* (6) *and denote by $a_n^{\hat{x}}(s)$ the Fourier transform of the function $x \to a_n^x(s)$. Then it holds that*

$$a_n^{\hat{x}}(s) = e^{-\frac{x^2 R(s)}{2}} \frac{(-i)^n x^n}{n!}. \tag{7}$$

Proof. Using formula (5) one can prove that the Fourier transform of $g(x) = \delta(x - B_s)$ admits the chaos expansion

$$\hat{g}(x) = \sum_{n\geqslant 0} a_n^{\hat{x}}(s) I_n\big(1_{[0,s]}^{\otimes n}\big) \tag{8}$$

where $a_n^{\hat{x}}(s)$ denotes the Fourier transform of the function $x \to a_n^x(s)$.

But on the other hand, we know that $\hat{\delta}(x - B_s) = e^{-ixB_s}$ and using Stroock's formula to decompose in chaos a square integrable random variable F infinitely differentiable in the Malliavin sense

$$F = \sum_{n\geqslant 0} \frac{1}{n!} I_n\big[\mathbf{E}\big(D^{(n)}F\big)\big]$$

where $D^{(n)}$ denotes the nth Malliavin derivative and by the trivial relation

$$D_u e^{-ixB_s} = -ixe^{-ixB_s} 1_{[0,s]}(u)$$

which implies

$$D^{(n)}_{u_1,\dots,u_n} e^{-ixB_s} = (-ix)^n e^{-ixB_s} 1^{\otimes n}_{[0,s]}(u_1,\dots,u_n)$$

one obtains

$$\hat{g}(x) = \mathbf{E}\bigl(e^{-ixB_s}\bigr) \sum_{n\geqslant 0} \frac{(-i)^n x^n}{n!} I_n\bigl(1^{\otimes n}_{[0,s]}\bigr) = e^{-\frac{x^2 R(s)}{2}} \sum_{n\geqslant 0} \frac{(-i)^n x^n}{n!} I_n\bigl(1^{\otimes n}_{[0,s]}\bigr). \tag{9}$$

Now, by putting together (8) and (9) we get (7). $\square$

4.1. The one-dimensional Brownian currents

Let B be in this part a the Wiener process. Then by (6)

$$\delta(x - B_s) = \sum_{n\geqslant 0} s^{-\frac{n}{2}} p_s(x) H_n\left(\frac{x}{s^{1/2}}\right) I_n\bigl(1^{\otimes n}_{[0,s]}\bigr) = \sum_{n\geqslant 0} a^x_n(s) I_n\bigl(1^{\otimes n}_{[0,s]}\bigr), \tag{10}$$

here I_n representing the multiple Wiener–Itô integral of degree n with respect to the Wiener process. In this case we have

$$a^x_n(s) = s^{-\frac{n}{2}} p_s(x) H_n\left(\frac{x}{s^{1/2}}\right).$$

We prove first the following result, which is already known (see [6,7]) but the method is different and it can also used to the fractional Brownian motion case and to the multidimensional context.

Proposition 1. *Let ξ be given by* (2)*. For every $x \in \mathbb{R}$, it holds that*

$$\mathbf{E}\bigl|\hat{\xi}(x)\bigr|^2 = T.$$

As a consequence $\xi \in H^{-r}(\mathbb{R},\mathbb{R})$ if and only if $r > \frac{1}{2}$.

Proof. We can write, by using the expression of the Skorohod integral via chaos expansion

$$\xi(x) = \sum_{n\geqslant 0} I_{n+1}\bigl(\bigl(a^x_n(s) 1^{\otimes n}_{[0,s]}(\cdot)\bigr)^{\sim}\bigr) \tag{11}$$

where $(a^x_n(s)1^{\otimes n}_{[0,s]}(\cdot))^{\sim}$ denotes the symmetrization in $n+1$ variables of the function $(s, t_1, \dots, t_n) \to a^x_n(s) 1^{\otimes n}_{[0,s]}(t_1,\dots,t_n)$. It can be shown that

$$\hat{\xi}(x) = \sum_{n\geqslant 0} I_{n+1}\bigl(\bigl(a^{\hat{x}}_n(s) 1^{\otimes n}_{[0,s]}(\cdot)\bigr)^{\sim}\bigr).$$

By using (7),

$$\hat{\xi}(x) = \sum_{n\geqslant 0} I_{n+1}\left(\left(\frac{(-i)^n x^n}{n!} e^{-\frac{x^2 s}{2}} 1_{[0,s]}^{\otimes n}(\cdot)\right)^{\sim}\right) = \sum_{n\geqslant 0} \frac{(-i)^n x^n}{n!} I_{n+1}\left(\left(e^{-\frac{x^2 s}{2}} 1_{[0,s]}^{\otimes n}(\cdot)\right)^{\sim}\right)$$

and therefore, by the isometry of multiple integrals (3),

$$\mathbf{E}\left|\hat{\xi}(x)\right|^2 = \sum_{n\geqslant 0} \frac{x^{2n}}{(n!)^2}(n+1)! \left\| \left(e^{-\frac{x^2 s}{2}} 1_{[0,s]}^{\otimes n}(\cdot)\right)^{\sim} \right\|^2_{L^2([0,T]^{n+1})}. \tag{12}$$

Since

$$\left(e^{-\frac{x^2 s}{2}} 1_{[0,s]}^{\otimes n}(\cdot)\right)^{\sim}(t_1, \dots, t_{n+1}) = \frac{1}{n+1} \sum_{i=1}^{n+1} e^{-\frac{x^2 t_i}{2}} 1_{[0,t_i]}^{\otimes n}(t_1, \dots, \hat{t_i}, \dots, t_{n+1})$$

where $\hat{t_i}$ means that the variable t_i is missing, we obtain

$$\begin{aligned}
\mathbf{E}\left|\hat{\xi}(x)\right|^2 &= \sum_{n\geqslant 0} \frac{x^{2n}}{(n+1)!} \left\| \sum_{i=0}^{n+1} e^{-\frac{x^2 t_i}{2}} 1_{[0,t_i]}^{\otimes n}(t_1, \dots, \hat{t_i}, \dots, t_{n+1}) \right\|^2_{L^2([0,T]^{n+1})} \\
&= \sum_{n\geqslant 0} \frac{x^{2n}}{n!} \int_0^T ds \int_{[0,T]^n} dt_1 \dots dt_n \left(e^{-\frac{x^2 s}{2}} 1_{[0,s]}^{\otimes n}(t_1, \dots, t_n)\right)^2 \\
&= \sum_{n\geqslant 0} \frac{x^{2n}}{n!} \int_0^T ds\, e^{-x^2 s} s^n\, ds = \int_0^T e^{-x^2 s} \sum_{n\geqslant 0} \frac{(x^s s)^n}{n!}\, ds = T.
\end{aligned}$$

Finally,

$$\mathbf{E}|\xi|^2_{H^{-r}(\mathbb{R};\mathbb{R})} = \int_{\mathbb{R}} \left(1+x^2\right)^{-r} \mathbf{E}\left|\hat{\xi}(x)\right|^2 dx = \int_{\mathbb{R}} \left(1+x^2\right)^{-r} T\, dx$$

and this is finite if and only if $r > \frac{1}{2}$. □

Remark 1. The above result gives actually a rigorous meaning of the formal calculation $E(\int_0^T \delta(x - B_s)\, dB_s)^2 = E \int_0^T |\delta(x - B_s)|^2\, ds = T$.

4.2. The multidimensional multiparameter Brownian currents

Let us consider now the multidimensional situation. In this part we will actually treat the regularity of the function

$$\int_{[0,T]^N} \delta(x - B_s)\, dB_s \tag{13}$$

where $x \in \mathbb{R}^d$ and $B = (B^1, \dots, B^d)$ is a d-dimensional Wiener sheet with parameter $t \in [0, T]^N$. Here the term (13) represents the vector given by

$$\xi(x) = \int_{[0,T]^N} \delta(x - B_s)\, dB_s = \left(\int_{[0,T]^N} \delta(x - B_s)\, dB_s^1, \dots, \int_{[0,T]^N} \delta(x - B_s)\, dB_s^d \right)$$
$$:= \big(\xi_1(x), \dots, \xi_d(x)\big)$$

for every $x \in \mathbb{R}^d$.

The method considered above based on the chaos expansion of the delta Dirac function has advantage that, in contrast with the approaches in [7] or [5], it can be immediately extended to time parameter in $\mathbb{R}^N$. Moreover, we are able to compute explicitly the Sobolev norm of the vector (13) and to obtain an "if and only if" result.

Proposition 2. *Let $(B_t)_{t\in[0,T]^N}$ be a d-dimensional Wiener process with multidimensional parameter $t \in [0, T]^N$. Then for every $w \in \Omega$ the mapping $x \to \int_{[0,T]^N} \delta(x - B_s)\, dB_s$ belongs to the negative Sobolev space $H^{-r}(\mathbb{R}^d, \mathbb{R}^d)$ if and only if $r > \frac{d}{2}$.*

Proof. We need to estimate

$$\int_{\mathbb{R}^d} \big(1 + |x|^2\big)^{-r} \mathbf{E}\big|\hat{\xi}(x)\big|^2\, dx$$

with $|\hat{\xi}(x)|^2 = |\hat{\xi}_1(x)|^2 + \cdots + |\hat{\xi}_d(x)|^2$ where $|\hat{\xi}_i(x)|$ denotes the complex modulus of $\hat{\xi}_i(x) \in \mathbb{C}$.

We can formally write (but it can also written in a rigorous manner by approximating the delta Dirac function by Gaussian kernels with variance $\varepsilon \to 0$)

$$\delta(x - B_s) = \prod_{j=1}^{d} \delta\big(x_j - B_s^j\big) = \delta_k(x - B_s)\delta\big(x_k - B_s^k\big)$$

where we denoted

$$\delta_k(x - B_s) = \prod_{j=1, j\neq k}^{d} \delta\big(x_j - B_s^j\big), \quad k = 1, \dots, d. \tag{14}$$

Let us compute the kth component of the vector $\xi(x)$. We will use the chaotic expansion for the delta Dirac function with multidimensional parameter $s \in [0, T]^N$ (see (5))

$$\delta\big(x_j - B_s^j\big) = \sum_{n_j \geqslant 0} a_{n_j}^{x_j}(s) I_{n_j}^j \big(1_{[0,s]}^{\otimes n_j}(\cdot)\big) \tag{15}$$

where I_n^j denotes the Wiener–Itô integral of order n with respect to the component B^j and for $s = (s_1, \dots, s_N)$

$$a_{n_j}^{x_j}(s) = p_{s_1\dots s_N}(x_j) H_{n_j}\left(\frac{x_j}{\sqrt{s_1\dots s_N}}\right)(s_1\dots s_N)^{-\frac{n_j}{2}}. \tag{16}$$

It holds, for every $k = 1, \dots, d$, by using formula (10)

$$\begin{aligned}
\xi_k(x) = \int_{[0,T]^N} \delta(x - B_s)\, dB_s^k &= \int_{[0,T]^N} \delta_k(x - B_s)\delta\big(x_k - B_s^k\big)\, dB_s^k \\
&= \int_{[0,T]^N} \delta_k(x - B_s) \sum_{n_k \geqslant 0} a_{n_k}^{x_k}(s) I_{n_k}^k\big(1_{[0,s]}^{\otimes n_k}(\cdot)\big)\, dB_s^k.
\end{aligned}$$

Since the components of the Brownian motion B are independent, the term $\delta_k(x - B_s)$ is viewed as a deterministic function when we integrate with respect to B^k. We obtain,

$$\xi_k(x) = \int_{[0,T]^N} \delta(x - B_s)\, dB_s^k = \sum_{n_k \geqslant 0} I_{n_k+1}\Big(\big(\delta_k(x - B_s) a_{n_k}^{x_k}(s) 1_{[0,s]}^{\otimes n_k}(\cdot)\big)^{\sim}\Big).$$

Here $(\delta_k(x - B_s) a_{n_k}^{x_k}(s) 1_{[0,s]}^{\otimes n_k}(\cdot))^{\sim}$ denotes the symmetrization in $n_k + 1$ variables of the function

$$(s, t_1, \dots, t_{n_k}) = \delta_k(x - B_s) a_{n_k}^{x_k}(s) 1_{[0,s]}^{\otimes n_k}(t_1, \dots, t_{n_k}).$$

Let us denote by $\hat{\xi}_k(x)$ the Fourier transform of $\xi_k(x) = \int_{[0,T]^N} \delta(x - B_s)\, dB_s^k$. As in the previous section one can show that (that is, the Fourier transform with respect to x "goes inside" the stochastic integral)

$$\hat{\xi}_k(x) = \sum_{n_k \geqslant 0} I_{n_k+1}^k\Big(\big(\delta_k(\hat{x} - B_s) a_{n_k}^{\hat{x}_k}(s) 1_{[0,s]}^{\otimes n_k}(\cdot)\big)^{\sim}\Big) \tag{17}$$

where

$$\delta_k(\hat{x} - B_s) a_{n_k}^{\hat{x}_k}(s)$$

is the Fourier transform of

$$\mathbb{R}^d \ni x = (x_1, \dots, x_d) \to \delta_k(x - B_s) a_{n_k}^{x_k}(s) = \left(\prod_{j=1, j\neq k}^{d} \delta\big(x_j - B_s^j\big)\right) a_{n_k}^{x_k}(s).$$

Now clearly

$$\delta_k(\hat{x} - B_s) = e^{-i\sum_{l=1, l\neq k}^{d} x_l B_s^l}$$

and by (7)

$$a_{n_k}^{\hat{x}_k}(s) = \frac{(-i)^{n_k} x_k^{n_k}}{n_k!} \mathbf{E}\big(e^{-i x_k B_s^k}\big) = \frac{(-i)^{n_k} x_k^{n_k}}{n_k!} e^{\frac{x_k^2 |s|}{2}}$$

with $|s| = s_1 \dots s_N$ if $s = (s_1, \dots, s_N)$. Thus relation (17) becomes

$$\hat{\xi}_k(x) = \sum_{n_k \geqslant 0} I_{n_k+1}^k \left(\left(e^{-i \sum_{l=1, l\neq k}^d x_l B_s^l} \frac{(-i)^{n_k} x_k^{n_k}}{n_k!} \mathbf{E}\left(e^{-i x_k B_s^k}\right) 1_{[0,s]}^{\otimes n_k}(\cdot) \right)^{\sim} \right). \tag{18}$$

Taking the modulus in $\mathbb{C}$ of the above expression, we get

$$\begin{aligned}
\left|\hat{\xi}_k(x)\right|^2 &= \left| \sum_{n_k \geqslant 0} I_{n_k+1}^k \left(\left(\cos\left(\sum_{l=1, l\neq k}^d x_l B_s^l \right) \frac{(-i)^{n_k} x_k^{n_k}}{n_k!} e^{-\frac{x_k^2 |s|}{2}} 1_{[0,s]}^{\otimes n_k}(\cdot) \right)^{\sim} \right) \right|^2 \\
&\quad + \left| \sum_{n_k \geqslant 0} I_{n_k+1}^k \left(\left(\sin\left(\sum_{l=1, l\neq k}^d x_l B_s^l \right) \frac{(-i)^{n_k} x_k^{n_k}}{n_k!} e^{-\frac{x_k^2 |s|}{2}} 1_{[0,s]}^{\otimes n_k}(\cdot) \right)^{\sim} \right) \right|^2
\end{aligned}$$

and using the isometry of multiple stochastic integrals (3) and the independence of components we get as in the proof of Proposition 1 (we use the notation $|s| = s_1 \dots s_N$)

$$\begin{aligned}
\mathbf{E}\left|\hat{\xi}_k(x)\right|^2 &= \sum_{n_k \geqslant 0} \frac{x_k^{2n_k}}{n_k!} \left\| \cos\left(\sum_{l=1, l\neq k}^d x_l B_s^l \right) e^{-\frac{x_k^2 |s|}{2}} 1_{[0,s]}^{\otimes n_k}(\cdot) \right\|_{L^2([0,T]^{N(n_k+1)})}^2 \\
&\quad + \sum_{n_k \geqslant 0} \frac{x_k^{2n_k}}{n_k!} \left\| \sin\left(\sum_{l=1, l\neq k}^d x_l B_s^l \right) e^{-\frac{x_k^2 |s|}{2}} 1_{[0,s]}^{\otimes n_k}(\cdot) \right\|_{L^2([0,T]^{N(n_k+1)})}^2 \\
&= \sum_{n_k \geqslant 0} \frac{x_k^{2n_k}}{n_k!} \int_{[0,T]^N} e^{-x_k^2 |s|} |s|^{n_k} \, ds \\
&= \int_{[0,T]^N} e^{-x_k^2 |s|} \left(\sum_{n_k \geqslant 0} \frac{(x_k^2 |s|)^{n_k}}{n_k!} \right) ds = \int_{[0,T]^N} ds = T^N.
\end{aligned}$$

This implies

$$\begin{aligned}
\int_{\mathbb{R}^d} \left(1 + |x|^2\right)^{-r} \mathbf{E}\left|\hat{\xi}(x)\right|^2 dx &= \int_{\mathbb{R}^d} \left(1 + |x|^2\right)^{-r} \mathbf{E}\left(\left|\hat{\xi}_1(x)\right|^2 + \dots + \left|\hat{\xi}_d(x)\right|^2\right) dx \\
&= d T^N \int_{\mathbb{R}^d} \left(1 + |x|^2\right)^{-r} dx
\end{aligned}$$

which is finite if and only if $2r > d$. $\square$

Remark 2. It is interesting to observe that the dimension N of the time parameter does not affect the regularity of currents. This is somehow unexpected because it is known that this dimension N influences the regularity of the local time of the process B which can be formally written as $\int_{[0,T]^N} \delta(x - B_s)\, ds$ (see [12]).

5. Regularity of fractional currents with respect to x

5.1. The one-dimensional fractional Brownian currents

We will consider in this paragraph a fractional Brownian motion $(B_t^H)_{t\in[0,T]}$ with Hurst parameter $H \in (\frac{1}{2}, 1)$. That is, B^H is a centered Gaussian process starting from zero with covariance function

$$R^H(t,s) := \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \quad s,t \in [0,T].$$

Let us denote by $\mathcal{H}_H$ the canonical Hilbert space of the fractional Brownian motion which is defined as the closure of the linear space generated by the indicator functions $\{1_{[0,t]},\ t \in [0,T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}_H} = R^H(t,s), \quad s,t \in [0,T].$$

One can construct multiple integrals with respect to B^H (the underlying space $L^2[0,T]$ is replaced by $\mathcal{H}_H$) and these integrals are those who appear in the formula (10). We prove the following result.

Proposition 3. *Let B^H be a fractional Brownian motion with $H \in (\frac{1}{2}, 1)$ and let ξ be given by* (2)*. Then for every $w \in \Omega$, we have*

$$\xi \in H^{-r}(\mathbb{R};\mathbb{R})$$

for every $r > \frac{1}{2H} - \frac{1}{2}$.

Proof. Using the above computations, we will get that the Fourier transform of $g(x) = \delta(x - B_s^H)$ is equal, on one hand, to

$$\hat{g}(x) = \sum_{n\geqslant 0} a_n^{\hat{x}}\big(s^{2H}\big) I_n^{B^H}\big(1_{[0,t]}^{\otimes n}(\cdot)\big)$$

(here $I_n^{B^H}$ denotes the multiple stochastic integral with respect to the fBm B^H and the function a_n^x is defined by (6)), and on the other hand,

$$\hat{g}(x) = \mathbf{E}\big(e^{-ixB_s^H}\big) \sum_{n\geqslant 0} \frac{(-ix)^n}{n!} I_n^B\big(1_{[0,t]}^{\otimes n}(\cdot)\big)$$

and so

$$a_n^{\hat{x}}\big(s^{2H}\big) = e^{-\frac{x^2 s^{2H}}{2}} \frac{(-ix)^n}{n!}. \tag{19}$$

Moreover

$$\hat{\xi}(x) = \sum_{n\geqslant 0} \frac{(-i)^n x^n}{n!} I_{n+1}^{B^H}\left(\left(e^{-\frac{x^2 s^{2H}}{2}} 1_{[0,s]}^{\otimes n}(\cdot)\right)^{\sim}\right).$$

We then obtain

$$\begin{aligned}
\mathbf{E}\|\xi\|^2_{H^{-r}(\mathbb{R};\mathbb{R})} &= \mathbf{E}\int_{\mathbb{R}} \frac{1}{(1+x^2)^r} \left|\hat{\xi}(x)\right|^2 dx \\
&= \int_{\mathbb{R}} \frac{1}{(1+x^2)^r} \sum_{n\geqslant 0} \frac{x^{2n}}{(n!)^2}(n+1)! \left\| \left(e^{-\frac{x^2 s^{2H}}{2}} 1_{[0,s]}^{\otimes n}(\cdot)\right)^{\sim} \right\|^2_{\mathcal{H}^{\otimes n+1}} dx \\
&= \int_{\mathbb{R}} \frac{1}{(1+x^2)^r} \sum_{n\geqslant 0} \frac{x^{2n}}{(n+1)!} dx \Bigg\langle \sum_{i=1}^{n+1} e^{-\frac{x^2 u_i^{2H}}{2}} 1_{[0,u_i]}^{\otimes n}(u_1,\dots,\hat{u}_i,\dots,u_{n+1}), \\
&\quad \sum_{j=1}^{n+1} e^{-\frac{x^2 v_j^{2H}}{2}} 1_{[0,v_j]}^{\otimes n}(v_1,\dots,\hat{v}_j,\dots,v_{n+1}) \Bigg\rangle_{\mathcal{H}_H}.
\end{aligned}$$

Using the fact that for regular enough functions f and g in $\mathcal{H}_H$ their product scalar is given by (see e.g. [21], Chapter 5) $\langle f,g\rangle_{\mathcal{H}_H} = H(2H-1)\int_0^T\int_0^T f(x)g(y)|x-y\beta|^{2H-2}\,dx\,dy$ and thus for regular enough function $f,g\in\mathcal{H}^{\otimes n}$

$$\begin{aligned}
\langle f,g\rangle_{\mathcal{H}_H^{\otimes n}} = \big(H(2H-1)\big)^n &\int_{[0,T]^n}\int_{[0,T]^n} f(u_1,\dots,u_n)g(v_1,\dots,v_n) \\
&\times \prod_{i=1}^n |u_i-v_i|^{2H-2}\,du_1\dots du_n\,dv_1\dots dv_n
\end{aligned}$$

we find (by $du_i\,du_j$ we mean below $du_1\dots du_{n+1}\,dv_1\dots dv_{n+1}$)

$$\begin{aligned}
&\mathbf{E}\|\xi\|^2_{H^{-r}(\mathbb{R};\mathbb{R})} \\
&\quad = \int_{\mathbb{R}} \frac{1}{(1+x^2)^r} \sum_{n\geqslant 0}\big(H(2H-1)\big)^{n+1} \frac{x^{2n}}{(n+1)!} dx \\
&\qquad \times \sum_{i,j=1}^{n+1} \int_{[0,T]^{n+1}}\int_{[0,T]^{n+1}} \prod_{l=1}^{n+1} |u_l-v_l|^{2H-2}\,du_i\,dv_j \\
&\qquad \times e^{-\frac{x^2 u_i^{2H}}{2}} e^{-\frac{x^2 v_j^{2H}}{2}} 1_{[0,u_i]}^{\otimes n}(u_1,\dots,\hat{u}_i,\dots,u_{n+1}) 1_{[0,v_j]}^{\otimes n}(v_1,\dots,\hat{v}_j,\dots,v_{n+1}) \\
&\quad = \int_{\mathbb{R}} \frac{1}{(1+x^2)^r} \sum_{n\geqslant 0}\big(H(2H-1)\big)^{n+1} \frac{x^{2n}}{(n+1)!} dx \\
&\qquad \times \sum_{i=1}^{n+1} \int_{[0,T]^{n+1}}\int_{[0,T]^{n+1}} e^{-\frac{x^2 u_i^{2H}}{2}} e^{-\frac{x^2 v_i^{2H}}{2}} \prod_{l=1}^{n+1} |u_l-v_l|^{2H-2}\,du_i\,dv_j
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}} \frac{1}{(1+x^2)^r} \sum_{n\geqslant 1} \big(2H(2H-1)\big)^{n+1} \frac{x^{2n}}{(n+1)!}\, dx \\
&\times \sum_{i,j=1;\, i\neq j}^{n+1} \int_{[0,T]^{n+1}} \int_{[0,T]^{n+1}} \prod_{l=1}^{n+1} |u_l - v_l|^{2H-2}\, du_i\, dv_j \\
&\times e^{-\frac{x^2 u_i^{2H}}{2}} e^{-\frac{x^2 v_j^{2H}}{2}} 1_{[0,u_i]}^{\otimes n}(u_1, \ldots, \hat{u}_i, \ldots, u_{n+1}) 1_{[0,v_j]}^{\otimes n}(v_1, \ldots, \hat{v}_j, \ldots, v_{n+1}) \\
:={}& A + B.
\end{aligned}
$$

Let us compute first the term A. Using the symmetry of the integrand and the fact that

$$
2H(2H-1) \int_0^t \int_0^s |u-v|^{2H-2}\, du\, dv = R(t,s), \quad s,t \in [0,T] \tag{20}
$$

we can write

$$
\begin{aligned}
A = {}& \int_{\mathbb{R}} \frac{1}{(1+x^2)^r} \sum_{n\geqslant 0} \big(2H(2H-1)\big)^{n+1} \frac{x^{2n}}{n!}\, dx \\
&\times \sum_{i=1}^{n+1} \int_{[0,T]^{n+1}} \int_{[0,T]^{n+1}} \prod_{l=1}^{n+1} |u_l - v_l|^{2H-2}\, du_i\, dv_j \\
&\times e^{-\frac{x^2 u_1^{2H}}{2}} e^{-\frac{x^2 v_1^{2H}}{2}} 1_{[0,u_1]}^{\otimes n}(u_2, \ldots, u_{n+1}) 1_{[0,v_1]}^{\otimes n}(v_2, \ldots, v_{n+1})
\end{aligned}
$$

and by integrating with respect to $du_2 \ldots du_{n+1}$

$$
\begin{aligned}
A = {}& \int_{\mathbb{R}} dx \frac{1}{(1+x^2)^r} \sum_{n\geqslant 0} 2H(2H-1) \frac{x^{2n}}{n!}\, dx \\
&\times \int_0^T \int_0^T e^{-\frac{x^2 u^{2H}}{2}} e^{-\frac{x^2 v^{2H}}{2}} |u-v|^{2H-2} R(u,v)^n\, du\, dv \\
={}& H(2H-1) \int_{\mathbb{R}} dx \frac{1}{(1+x^2)^r} \int_0^T \int_0^T e^{-\frac{x^2 u^{2H}}{2}} e^{-\frac{x^2 v^{2H}}{2}} |u-v|^{2H-2} \bigg(\sum_{n\geqslant 0} \frac{x^{2n} R(u,v)^n}{n!} \bigg) du\, dv \\
={}& H(2H-1) \int_{\mathbb{R}} dx \frac{1}{(1+x^2)^r} \int_0^T \int_0^T e^{-\frac{x^2 u^{2H}}{2}} e^{-\frac{x^2 v^{2H}}{2}} |u-v|^{2H-2} e^{x^2 R(u,v)}\, du\, dv \\
={}& H(2H-1) \int_{\mathbb{R}} \frac{1}{(1+x^2)^r} \int_0^T \int_0^T e^{-\frac{x^2 |u-v|^{2H}}{2}} |u-v|^{2H-2} du\, dv.
\end{aligned}
$$

Now, by classical Fubini,

$$A = H(2H-1)\int_0^T\int_0^T |u-v|^{2H-2}\,du\,dv \int_{\mathbb{R}} \frac{1}{(1+x^2)^r} e^{-\frac{x^2|u-v|^{2H}}{2}}\,dx$$

and by using the change of variable $x|u-v|^H = y$ in the integral with respect to,

$$A = H(2H-1)\int_0^T\int_0^T |u-v|^{2H-2+2Hr-H} \int_{\mathbb{R}} e^{-\frac{y^2}{2}} \left(|u-v|^{2H}+y^2\right)^{-r}\,dy$$

and this is finite when $2H-2+2Hr-H > -1$ which gives $r > \frac{1}{2H} - \frac{1}{2}$.

The term denoted by B can be treated as follows

$$\begin{aligned} B &= \int_{\mathbb{R}} \frac{1}{(1+x^2)^r} \sum_{n\geqslant 1} \frac{x^{2n}}{(n+1)!} n(n+1)\big(2H(2H-1)\big)^{n+1} \int_{[0,T]^{n+1}} \int_{[0,T]^{n+1}} du_i\,dv_j \\ &\quad \times e^{-\frac{x^2u_1^{2H}}{2}} e^{-\frac{x^2v_2^{2H}}{2}} 1_{[0,u_1]}(u_2,\ldots,u_{n+1})^{\otimes n} 1_{[0,v_2]}(v_1,v_3,\ldots,v_{n+1})^{\otimes n} \prod_{l=1}^{n+1} |u_l-v_l|^{2H-2} \\ &= \big(2H(2H-1)\big)^2 \int_{\mathbb{R}} \frac{1}{(1+x^2)^r} \sum_{n\geqslant 1} \frac{x^{2n}}{(n-1)!} \int_0^T\int_0^T\int_0^T\int_0^T du_1\,du_2\,dv_1\,dv_2 \\ &\quad \times e^{-\frac{x^2u_1^{2H}}{2}} e^{-\frac{x^2v_2^{2H}}{2}} 1_{[0,u_1]}(u_2) 1_{[0,v_2]}(v_1) R(u_1,v_2)^{n-1} |u_1-v_1|^{2H-2} |u_2-v_2|^{2H-2}. \end{aligned}$$

Since

$$\sum_{n\geqslant 1} \frac{x^{2n}}{(n-1)!} R(u_1,v_2)^{n-1} = x^2 \sum_{n\geqslant 0} \frac{x^{2n}R(u_1,v_2)^n}{n!} = x^2 e^{x^2R(u_1,v_2)}$$

we obtain

$$\begin{aligned} B &= \big(2H(2H-1)\big)^2 \int_{\mathbb{R}} \frac{1}{(1+x^2)^r} x^2 \int_0^T\int_0^T\int_0^T\int_0^T du_1\,du_2\,dv_1\,dv_2 \\ &\quad \times e^{-\frac{x^2|u_1-v_2|^{2H}}{2}} |u_1-v_1|^{2H-2} |u_2-v_2|^{2H-2} 1_{[0,u_1]}(u_2) 1_{[0,v_2]}(v_1). \end{aligned}$$

We calculate first the integral du_2 and dv_1 and then in dx we use the change of variables $x|u_1-v_2|^H = y$ and we get

$$
\begin{aligned}
B &= H^2 \int_{\mathbb{R}} \frac{1}{(1+x^2)^r} x^2 \int_0^T \int_0^T du_1\, dv_2\, e^{-\frac{x^2|u_1-v_2|^{2H}}{2}} \\
&\quad \times \left(|u_1-v_2|^{2H-1} - u_1^{2H-1}\right)\left(|u_1-v_2|^{2H-1} - v_2^{2H-1}\right) \\
&= \int_0^T du_1\, dv_2 \left(|u_1-v_2|^{2H-1} - u_1^{2H-1}\right)\left(|u_1-v_2|^{2H-1} - v_2^{2H-1}\right)|u_1-v_2|^{2Hr-2H-H} \\
&\quad \times \int_{\mathbb{R}} y^2 e^{-\frac{y^2}{2}} \left(|u_1-v_2|^{2H} + y^2\right) dy
\end{aligned}
$$

which is finite if $4H-2+2Hr-3H > -1$ and this implies again $r > \frac{1}{2H} - \frac{1}{2}$. $\square$

5.2. The multidimensional fractional currents

We will also consider the multidimensional case of the fractional Brownian motion B^H in $\mathbb{R}^d$. It is defined as a random vector $B^H = (B^{H_1}, \dots, B^{H_d})$ where B^{H_i} are independent one-dimensional fractional Brownian sheets. We will assume that $H_i \in (\frac{1}{2}, 1)$ for every $i = 1, \dots, d$. In this case

$$
\xi(x) = \left(\int_{[0,T]^N} \delta\left(x - B_s^H\right) dB^{H_1}, \dots, \int_{[0,T]^N} \delta\left(x - B_s^H\right) dB^{H_d} \right)
$$

where the above integral is a divergence (Skorohod) integral with respect to the fractional Brownian motion. We mention that the canonical Hilbert space $\mathcal{H}_{H_k}$ of the fractional Brownian sheet B^{H_k} is now the closure of the linear space of the indicator functions with respect to the inner product

$$
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}_{H_k}} = \mathbf{E}\left(B_t^{H_k} B_s^{H_k}\right) := R^{H_k}(t,s) = \prod_{i=1}^N R^{H_k}(t_i, s_i)
$$

if $t = (t_1, \dots, t_N)$ and $s = (s_1, \dots, s_N)$. In this case we have

Proposition 4. *Let B^H be as above. Then for every ω the fractional Brownian current ξ belong to the Sobolev space $H^{-r}(\mathbb{R}^d; \mathbb{R}^d)$ for every $r > \max_{k=1,\dots,d}(\frac{d}{2} - 1 + \frac{1}{2H_k})$.*

Proof. We write

$$
\begin{aligned}
\xi_k(x) = \int_{[0,T]^N} \delta\left(x - B_s^H\right) dB_s^{H_k} &= \int_{[0,T]^N} \delta_k\left(x - B_s^H\right) \delta\left(x_k - B_s^{H_k}\right) dB_s^{H_k} \\
&= \int_{[0,T]^N} \delta_k\left(x - B_s^H\right) \sum_{n_k \geqslant 0} a_{n_k}^{x_k}(s) I_{n_k}^{H_k}\left(1_{[0,s]}^{\otimes n_k}(\cdot)\right) dB_s^{H_k}
\end{aligned}
$$

and

$$\hat{\xi}_k(x) = \sum_{n_k \geqslant 0} I^{H_k}_{n_k+1}\left(\left(e^{-i\sum_{l=1,l\neq k}^d x_l B_s^{H_l}} \frac{(-i)^{n_k} x_k^{n_k}}{n_k!} e^{-\frac{x_2|s|^{2H_k}}{2}} 1^{\otimes n_k}_{[0,s]}(\cdot)\right)^{\sim}\right).$$

Here again the integral dB^{H_k} denotes the Skorohod integral with respect to the fractional Brownian sheet B^{H_k} and I^{H_k} denotes the multiple integral with respect to B^{H_k}. Then

$$\begin{aligned}\left|\hat{\xi}_k(x)\right|^2 = {} & \left|\sum_{n_k \geqslant 0} I^{H_k}_{n_k+1}\left(\left(\cos\left(\sum_{l=1,\, l\neq k}^d x_l B_s^{H_l}\right)\frac{x_k^{n_k}}{n_k!} e^{-\frac{x^2|s|^{2H_k}}{2}} 1^{\otimes n_k}_{[0,s]}(\cdot)\right)^{\sim}\right)\right|^2 \\ & + \left|\sum_{n_k \geqslant 0} I^{H_k}_{n_k+1}\left(\left(\sin\left(\sum_{l=1,\, l\neq k}^d x_l B_s^{H_l}\right)\frac{x_k^{n_k}}{n_k!} e^{-\frac{x^2|s|^{2H_k}}{2}} 1^{\otimes n_k}_{[0,s]}(\cdot)\right)^{\sim}\right)\right|^2\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}\left|\hat{\xi}_k(x)\right|^2 = {} & \sum_{n_k \geqslant 0}(n_k+1)!\frac{x_k^{2n_k}}{(n_k!)^2}\left\|\left(\cos\left(\sum_{l=1,\, l\neq k}^d x_l B_s^{H_l}\right) e^{-\frac{x^2|s|^{2H_k}}{2}} 1^{\otimes n_k}_{[0,s]}(\cdot)\right)^{\sim}\right\|^2_{(\mathcal{H}_{H_k})^{\otimes(n_k+1)}} \\ & + \sum_{n_k \geqslant 0}(n_k+1)!\frac{x_k^{2n_k}}{(n_k!)^2}\left\|\left(\sin\left(\sum_{l=1,\, l\neq k}^d x_l B_s^{H_l}\right) e^{-\frac{x^2|s|^{2H_k}}{2}} 1^{\otimes n_k}_{[0,s]}(\cdot)\right)^{\sim}\right\|^2_{(\mathcal{H}_{H_k})^{\otimes(n_k+1)}}.\end{aligned}$$

Note that, if f, g are two regular functions of N variables

$$\begin{aligned}\langle f, g\rangle_{\mathcal{H}_{H_k}} &= \big(H_k(2H_k-1)\big)^N \int_{[0,T]^N}\int_{[0,T]^N} f(u)g(u)\prod_{i=1}^N |u_i - v_i|^{2H_k-2}\,du\,dv \\ &:= \big(H_k(2H_k-1)\big)^N \int_{[0,T]^N}\int_{[0,T]^N} f(u)g(v)\|u-v\|^{2H_k-2}\,du\,dv\end{aligned}$$

if $u = (u_1, \dots, u_N)$ and $v = (v_1, \dots, v_N)$ and if F, G are regular functions of $N(n_k+1)$ variables,

$$\begin{aligned}&\langle F, G\rangle_{(\mathcal{H}_{H_k})^{\otimes(n_k+1)}} \\ &\quad = \prod_{k=1}^{n_k+1}\big(H_k(2H_k-1)\big)^N \int_{[0,T]^{N(n_k+1)}}\int_{[0,T]^{N(n_k+1)}} F\big(u^1, \dots, u^{n_k+1}\big)G\big(v^1, \dots, v^{n_k+1}\big) \\ &\qquad \times \prod_{i=1}^{n_k+1}\left\|u^i - v^i\right\|^{2H_k-2} du^1 \dots du^{n_k+1}\,dv^1 \dots dv^{n_k+1}\end{aligned}$$

and thus, by symmetrizing the above function and taking the scalar product in $(\mathcal{H}_{H_k})^{\otimes(n_k+1)}$

$$
\begin{aligned}
&\mathbf{E}\left|\hat{\xi}_k(x)\right|^2\\
&\quad=\sum_{n_k\geqslant 0}\frac{x_k^{2n_k}}{(n_k+1)!}\big(H_k(2H_K-1)\big)^{N(n_k+1)}\\
&\qquad\times\sum_{i,j=1}^{n_k+1}\int\limits_{[0,T]^{N(n_k+1)}}du\int\limits_{[0,T]^{N(n_k+1)}}dv\prod_{l=1}^{n_k+1}\left\|u^l-v^l\right\|^{2H_k-2}\\
&\qquad\times\cos\left(\sum_{l=1,\,l\neq k}^{d}x_l B^{H_l}_{u^i}\right)\cos\left(\sum_{l=1,\,l\neq k}^{d}x_l B^{H_l}_{v^j}\right)e^{-\frac{x_k^2|u^i|^{2H_k}}{2}}e^{-\frac{x_k^2|v^j|^{2H_k}}{2}}\\
&\qquad\times 1^{\otimes n_k}_{[0,u^i]}\big(u^1,\dots,\hat{u}^i,\dots,u^{n_k+1}\big)1^{\otimes n_k}_{[0,v^j]}\big(v^1,\dots,\hat{v}^j,\dots,v^{n_k+1}\big)\\
&\qquad+\sum_{n_k\geqslant 0}\frac{x_k^{2n_k}}{(n_k+1)!}\big(H_k(2H_K-1)\big)^{N(n_k+1)}\\
&\qquad\times\sum_{i,j=1}^{n_k+1}\int\limits_{[0,T]^{N(n_k+1)}}du\int\limits_{[0,T]^{N(n_k+1)}}dv\prod_{l=1}^{n_k+1}\left\|u^l-v^l\right\|^{2H_k-2}\\
&\qquad\times\sin\left(\sum_{l=1,\,l\neq k}^{d}x_l B^{H_l}_{u^i}\right)\sin\left(\sum_{l=1,\,l\neq k}^{d}x_l B^{H_l}_{v^j}\right)e^{-\frac{x_k^2|u^i|^{2H_k}}{2}}e^{-\frac{x_k^2|v^j|^{2H_k}}{2}}\\
&\qquad\times 1^{\otimes n_k}_{[0,u^i]}\big(u^1,\dots,\hat{u}^i,\dots,u^{n_k+1}\big)1^{\otimes n_k}_{[0,v^j]}\big(v^1,\dots,\hat{v}^j,\dots,v^{n_k+1}\big).
\end{aligned}
$$

Here $|u^i|^{2H_k}=(u^i_1\dots u^i_N)^{2H_k}$ and $1_{[0,u^i]}=1_{[0,u^i_1]}\dots 1_{[0,u^i_N]}$ if $u^i=(u^i_1,\dots,u^i_N)$. The next step is to majorize $|(\cos(u)\cos(v)|$ by $\frac{1}{2}(\cos^2(u)+\cos^2(v))$ and similarly for the sinus.

$$
\begin{aligned}
&\mathbf{E}\left|\hat{\xi}_k(x)\right|^2\\
&\quad\leqslant cst.\sum_{n_k\geqslant 0}\frac{x_k^{2n_k}}{(n_k+1)!}\big(H_k(2H_K-1)\big)^{N(n_k+1)}\\
&\qquad\times\sum_{i=1}^{n_k+1}\int\limits_{[0,T]^{N(n_k+1)}}du\int\limits_{[0,T]^{N(n_k+1)}}dv\prod_{l=1}^{n_k+1}\left\|u^l-v^l\right\|^{2H_k-2}\\
&\qquad\times e^{-\frac{x^2|u^i|^{2H_k}}{2}}e^{-\frac{x^2|v^i|^{2H_k}}{2}}1^{\otimes n_k}_{[0,u^i]}\big(u^1,\dots,\hat{u}^i,\dots,u^{n_k+1}\big)1^{\otimes n_k}_{[0,v^j]}\big(v^1,\dots,\hat{v}^j,\dots,v^{n_k+1}\big)\\
&\qquad+cst.\sum_{n_k\geqslant 0}\frac{x_k^{2n_k}}{(n_k+1)!}\big(H_k(2H_K-1)\big)^{N(n_k+1)}\\
&\qquad\times\sum_{i,j=1;\,i\neq j}^{n_k+1}\int\limits_{[0,T]^{N(n_k+1)}}du\int\limits_{[0,T]^{N(n_k+1)}}dv\prod_{l=1}^{n_k+1}\left\|u^l-v^l\right\|^{2H_k-2}
\end{aligned}
$$

$$\times e^{-\frac{x_k^2|u^i|^{2H_k}}{2}} e^{-\frac{x_k^2|v^j|^{2H_k}}{2}} 1_{[0,u^i]}^{\otimes n_k}\big(u^1,\dots,\hat{u}^i,\dots,u^{n_k+1}\big) 1_{[0,v^j]}^{\otimes n_k}\big(v^1,\dots,\hat{v}^j,\dots,v^{n_k+1}\big)$$
$$:= C_k + D_k$$

and the two terms above can be treated as the terms A and B from the one-dimensional case. For example the term denotes by C_k (we illustrate only this term because D_k is similar to the term B in the one-dimensional case) gives

$$\begin{aligned} C_k &= \sum_{n_k \geqslant 0} \frac{x_k^{2n_k}}{n_k!} \big(H_k(2H_K-1)\big)^{N(n_k+1)} \int\limits_{[0,T]^{N(n_k+1)}} \int\limits_{[0,T]^{N(n_k+1)}} \prod_{l=1}^{n_k+1} \big\|u^l - v^l\big\|^{2H_k-2} \\ &\quad \times e^{-\frac{x_k^2|u^1|^{2H_k}}{2}} e^{-\frac{x_k^2|v^1|^{2H_k}}{2}} 1_{[0,u^1]}^{\otimes n_k}\big(u^2,\dots,\dots,u^{n_k+1}\big) 1_{[0,v^1]}^{\otimes n_k}\big(v^2,\dots,\dots,v^{n_k+1}\big) \\ &= \sum_{n_k \geqslant 0} \frac{x_k^{2n_k}}{n_k!} \big(H_k(2H_k-1)\big)^N \\ &\quad \times \int\limits_{[0,T]^N} \int\limits_{[0,T]^N} e^{-\frac{x_k^2|u^1|^{2H_k}}{2}} e^{-\frac{x_k^2|v^1|^{2H_k}}{2}} \big\|u^1 - v^1\big\|^{2H_k-2} R^{H_k}\big(u^1,v^1\big)^{n_k}\, d^1\, dv^1 \\ &= \big(H_k(2H_k-1)\big)^N \int\limits_{[0,T]^N} \int\limits_{[0,T]^N} e^{x_k^2 R(u^1,v^1)} e^{-\frac{x_k^2|u^1|^{2H_k}}{2}} e^{-\frac{x_k^2|v^1|^{2H_k}}{2}} \big\|u^1 - v^1\big\|^{2H_k-2}\, d^1\, dv^1 \end{aligned}$$

with $R^{H_k}(u^1,v^1) = R^{H_k}(u_1^1,v_1^1)\dots R^{H_k}(u_N^1,v_N^1)$. But $R^{H_k}(u,v) - u^{2H_k} - v^{2H_k} = \mathbf{E}(B_u^{H_k}B_v^{H_k}) - \mathbf{E}(B_u^{H_k})^2 - \mathbf{E}(B_v^{H_k})^2 = -\mathbf{E}(B_u^{H_k} - B_v^{H_k})^2$ for any $u, v \in \mathbb{R}^n$ and this implies

$$\int\limits_{\mathbb{R}^d} dx\,\big(1+|x|^2\big)^{-r} C_k = cst. \int\limits_{\mathbb{R}^d} dx\,\big(1+|x|^2\big)^{-r} \int\limits_{[0,T]^N} \int\limits_{[0,T]^N} e^{-\frac{x_k^2 \mathbf{E}(B_u^{H_k} - B_v^{H_k})^2}{2}} \|u-v\|^{2H_k-2}\, du\, dv.$$

The case $N = 1$ can be easily handled. Indeed, in this case

$$\begin{aligned} \int\limits_{\mathbb{R}^d} dx\,\big(1+|x|^2\big)^{-r} C_k &= \int\limits_{[0,T]} \int\limits_{[0,T]} |u-v|^{2H_k-2} \int\limits_{\mathbb{R}^d} \big(1+|x|\big)^{-r} e^{-\frac{x_k^2|u-v|^{2H_k}}{2}}\, dx \\ &= \int\limits_{[0,T]} \int\limits_{[0,T]} |u-v|^{2H_k-2+2H_k r - dH_k} \int\limits_{\mathbb{R}^d} e^{-\frac{y_k^2}{2}} \big(|u-v|^{2H_k} + y^2\big)^{-r}\, dy \end{aligned}$$

which is finite if $r > \frac{1}{2H_k} + \frac{d}{2} - 1$. When $N \geqslant 2$ we will have

$$\int\limits_{\mathbb{R}^d} dx\big(1+|x|^2\big)^{-r} C_k = \int\limits_{[0,T]^N} \int\limits_{[0,T]^N} \|u-v\|^{2H_k-2} \big(\mathbf{E}\big(B_u^{H_k} - B_v^{H_k}\big)^2\big)^{d/2+r}$$

$$\times \int_{\mathbb{R}^d} e^{-y_k^2/2}\big[\mathbf{E}\big(B_u^{H_k} - B_v^{H_k}\big)^2 + y^2\big]^{-r}\,dx$$

and this is finite if $r > \frac{1}{2H_k} + \frac{d}{2} - 1$. $\square$

Remark 3. In [7] the authors obtained the same regularity in the case of the pathwise integral with respect to the fBm. They considered only the case $H_k = H$ for every $k = 1, \ldots, d$ and $N = 1$. On the other hand [7], the Hurst parameter is allowed to be lesser than one half, it is assumed to be in $(\frac{1}{4}, 1)$.

6. Regularity of stochastic currents with respect to ω

6.1. The Brownian case

We study now, for fixed $x \in \mathbb{R}$, the regularity of the functional ξ in the sense of Watanabe. We would like to see if, as for the delta Dirac function, the stochastic integral $\xi(x)$ keeps the same regularity in the Watanabe spaces (as a function of ω) and in the Sobolev spaces as a function of x. We consider B a one-dimensional Wiener process and denote

$$\xi(x) = \int_a^T \delta(x - B_s)\,dB_s$$

with $a > 0$. Recall that by (10)

$$\xi(x) = \sum_{n\geqslant 0} I_{n+1}\big(\big(a_n^x(s)1_{[0,s]}^{\otimes n}(\cdot)1_{[a,T]}(s)\big)^{\sim}\big)$$

(we consider the integral from $a > 0$ instead of zero to avoid a singularity) where

$$\big(a_n^x(s)1_{[0,s]}^{\otimes n}(\cdot)1_{[a,T]}(s)\big)^{\sim}(t_1, \ldots, t_{n+1}) = \frac{1}{n+1}\sum_{i=1}^{n+1} a_n^x(t_i)1_{[0,t_i]}^{\otimes n}(t_1, \ldots, \hat{t_i}, \ldots, t_{n+1})1_{[a,T]}(t_i)$$

where as above $\hat{t_i}$ means that the variable t_i is missing.

We recall that F is a random variable having the chaotic decomposition $F = \sum_n I_n^i(f_n)$ then its Sobolev–Watanabe norm is given by

$$\|F\|_{2,\alpha}^2 = \sum_n (n+1)^\alpha \big\|I_n(f_n)\big\|_{L^2(\Omega)}^2. \tag{21}$$

We will get

$$\begin{aligned}\big\|\xi(x)\big\|_{2,\alpha}^2 &= \sum_{n\geqslant 0}(n+2)^\alpha(n+1)!\big\|\big(a_n^x(s)1_{[0,s]}^{\otimes n}(\cdot)1_{[a,T]}(s)\big)^{\sim}\big\|_{L^2[0,T]^{n+1}}^2\\ &= \sum_{n_i\geqslant 0}(n+2)^\alpha n!\int_a^T\bigg(p_s(x)H_n\bigg(\frac{x}{\sqrt{s}}\bigg)\bigg)^2\,ds.\end{aligned}$$

We use the identity

$$H_n(y)e^{-\frac{y^2}{2}} = (-1)^{[n/2]}2^{\frac{n}{2}}\frac{2}{n!\pi}\int_0^\infty u^n e^{-u^2} g\big(uy\sqrt{2}\big)\,du \tag{22}$$

where $g(r) = \cos(r)$ if n is even and $g(r) = \sin(r)$ if n is odd. Since $|g(r)| \leqslant 1$, we have the bound

$$\big|H_n(y)e^{-y^2/2}\big| \leqslant 2^{\frac{n}{2}}\frac{2}{n!\pi}\Gamma\left(\frac{n+1}{2}\right) := c_n. \tag{23}$$

Then

$$\big\|\xi(x)\big\|_{2,\alpha}^2 \leqslant cst.\sum_n (n+2)^\alpha n! c_n^2 \int_a^T \frac{1}{s}\,ds \tag{24}$$

and this is finite for $\alpha < \frac{-1}{2}$ since by Stirling's formula $n!c_n^2$ behaves as $cst.\frac{1}{\sqrt{n}}$.

We summarize the above discussion.

Proposition 5. *For any $x \in \mathbb{R}$ the functional $\xi(x) = \int_0^T \delta(x - B_s)\,dB_s$ belongs to the Sobolev–Watanabe space $\mathbb{D}^{-\alpha,2}$ for any $\alpha > \frac{1}{2}$.*

Remark 4. As for the delta Dirac function, the regularity ξ is the same with respect to x and with respect to ω.

6.2. The fractional case

In this paragraph the driving process is a fractional Brownian motion $(B_t^H)_{t\in[0,T]}$ with Hurst parameter $H \in (\frac{1}{2}, 1)$. We are interested to study the regularity as a functional in Sobolev–Watanabe spaces of

$$\xi(x) = \int_0^T \delta\big(x - B_s^H\big)\,dB_s^H \tag{25}$$

when $x \in \mathbb{R}$ is fixed. We will that now the order of regularity in the Watanabe spaces changes and it differs from the order of regularity of the same functional with respect to the variable x.

We prove the following result.

Proposition 6. *Let $(B_t^H)_{t\in[0,T]}$ be a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. For any $x \in \mathbb{R}$ the functional $\xi(x)$ (2) is an element of the Sobolev–Watanabe space $\mathbb{D}^{-\alpha,2}$ with $\alpha > \frac{3}{2} - \frac{1}{2H}$.*

Proof. We will have in this case

$$\left\|\xi(x)\right\|_{2,\alpha}^{2}=\sum_{n\geqslant 0}(n+2)^{\alpha}(n+1)!\left\|\left(a_n^x(s)1_{[0,s]}^{\otimes n}(\cdot)\right)^{\sim}\right\|_{\mathcal{H}_H^{\otimes n+1}}^{2}$$

where we denoted by $\mathcal{H}_H$ the canonical Hilbert space of the fractional Brownian motion.

As in the proof of Proposition 2 we obtain

$$\begin{aligned}
&\left\|\left(a_n^x(s)1_{[0,s]}^{\otimes n}(\cdot)\right)^{\sim}\right\|_{\mathcal{H}_H^{\otimes n+1}}^{2}\\
&\quad=\frac{(H(2H-1))^{n+1}}{(n+1)^2}\sum_{i,j=1}^{n+1}\int\limits_{[0,T]^{n+1}}\int\limits_{[0,T]^{n+1}}du_1\dots du_{n+1}\,dv_1\dots dv_{n+1}\prod_{l=1}^{n+1}|u_l-v_l|^{2H-2}\\
&\qquad\times a_n^x(u_i)a_n^x(v_j)1_{[0,u_i]}^{\otimes n}(u_1,\dots,\hat{u}_i,\dots,u_{n+1})1_{[0,v_j]}^{\otimes n}(v_1,\dots,\hat{v}_j,\dots,v_{n+1})\\
&\quad=\frac{(H(2H-1))^{n+1}}{(n+1)^2}\sum_{i=1}^{n+1}\int\limits_{[0,T]^{n+1}}\int\limits_{[0,T]^{n+1}}du_1\dots du_{n+1}\,dv_1\dots dv_{n+1}\prod_{l=1}^{n+1}|u_l-v_l|^{2H-2}\\
&\qquad\times a_n^x(u_i)a_n^x(v_i)1_{[0,u_i]}^{\otimes n}(u_1,\dots,\hat{u}_i,\dots,u_{n+1})1_{[0,v_i]}^{\otimes n}(v_1,\dots,\hat{v}_i,\dots,v_{n+1})\\
&\qquad+\frac{(H(2H-1))^{n+1}}{(n+1)^2}\sum_{i\neq j;\,i,j=1}^{n+1}\int\limits_{[0,T]^{n+1}}\int\limits_{[0,T]^{n+1}}du_1\dots du_{n+1}\,dv_1\dots dv_{n+1}\prod_{l=1}^{n+1}|u_l-v_l|^{2H-2}\\
&\qquad\times a_n^x(u_i)a_n^x(v_j)1_{[0,u_i]}^{\otimes n}(u_1,\dots,\hat{u}_i,\dots,u_{n+1})1_{[0,v_j]}^{\otimes n}(v_1,\dots,\hat{v}_j,\dots,v_{n+1})\\
&\quad:=A(n)+B(n).
\end{aligned}$$

The first term $A(n)$ equals, by symmetry,

$$A(n)=\frac{(H(2H-1))^{n+1}}{n+1}\int_0^T\int_0^T du\,dv\,a_n^x(u)a_n^x(v)|u-v|^{2H-2}\left(\int_0^u\int_0^v|u'-v'|^{2H-2}\,du'\,dv'\right)^n$$

and using equality (20) we get

$$\begin{aligned}
A(n)={}&\frac{(H(2H-1))}{n+1}\int_0^T\int_0^T du\,dv\\
&\times|u-v|^{2H-2}R(u,v)^n u^{-Hn}v^{-Hn}p_{u^{2H}}(x)p_{v^{2H}}(x)H_n\left(\frac{x}{u^H}\right)H_n\left(\frac{x}{v^H}\right).
\end{aligned}$$

Using the identity (22)

$$
\begin{aligned}
&\sum_{n\geqslant 0}(n+2)^{\alpha}(n+1)!A(n)\\
&\quad = c(H)\sum_{n\geqslant 0}(n+2)^{\alpha}n!\int_0^T\int_0^T du\,dv\,|u-v|^{2H-2}\\
&\qquad \times R(u,v)^n u^{-Hn}v^{-Hn}p_{u^{2H}}(x)p_{v^{2H}}(x)H_n\left(\frac{x}{u^H}\right)H_n\left(\frac{x}{v^H}\right)\\
&\quad = c(H)\sum_{n\geqslant 0}(n+2)^{\alpha}n!c_n^2\int_0^T\int_0^T du\,dv\,|u-v|^{2H-2}R(u,v)^n u^{-Hn}v^{-Hn}u^{-H}v^{-H}.
\end{aligned}
$$

By the selfsimilarity of the fBm we have $R(u,v)=u^{-2H}R(1,\frac{v}{u})$ and by the change of variables $v=zu$ we obtain

$$
\begin{aligned}
&\sum_{n\geqslant 0}(n+2)^{\alpha}(n+1)!A(n)\\
&\qquad \leqslant c(H)\sum_{n\geqslant 0}(n+2)^{\alpha}n!c_n^2\left(\int_0^T u^{1-2H}\,du\right)\int_0^1 \frac{R(1,z)^n}{z^{Hn}}\frac{(1-z)^{2H-2}}{z^H}\,dz.
\end{aligned}
\tag{26}
$$

Using Lemma 2 (actually a slightly modification of it) in [3]

$$
\int_0^1 \frac{R(1,z)^n}{z^{Hn}}\frac{(1-z)^{2H-2}}{z^H}dz \leqslant c(H)\frac{1}{n^{\frac{1}{2H}}}
\tag{27}
$$

and the right-hand side of (26) is bounded, modulo a constant, by

$$
\sum_{n\geqslant 0}(n+2)^{\alpha}n!c_n^2 n^{-\frac{1}{2H}}
$$

and since $n!c_n^2$ behaves as $\frac{1}{\sqrt{n}}$, the last sum is convergent if $\frac{1}{2H}-\alpha+\frac{1}{2}>1$, or $-\alpha>-\frac{1}{2H}+\frac{1}{2}$.

Let us regard now the sum involving the term $B(n)$. It will actually decide the regularity of the functional $\xi(x)$. Following the computations contained in the proof of Proposition 3

$$
\begin{aligned}
&\sum_{n\geqslant 0}(n+2)^{\alpha}(n+1)!B(n)\\
&\quad = c(H)\sum_{n\geqslant 0}(n+2)^{\alpha}(n+1)!\frac{n(n+1)}{(n+1)^2}\int_0^T\int_0^T\int_0^T\int_0^T du_1\,du_2\,dv_1\,dv_2\\
&\qquad \times 1_{[0,u_1]}(u_2)1_{[0,v_2]}(v_1)R(u_1,v_2)^{n-1}|u_1-v_1|^{2H-2}|u_2-v_2|^{2H-2}a_n^x(u_1)a_n^x(v_2)
\end{aligned}
$$

$$\leqslant c(H)\sum_{n\geqslant 0}(n+2)^{\alpha}nn!c_n^2$$

$$\times\int_0^T\int_0^T du_1\,dv_2\,R(u_1,v_2)^{n-1}u_1^{-Hn}v_2^{-Hn}\left(\int_0^{u_1}|u_2-v_2|^{2H-2}\,du_2\right)\left(\int_0^{v_2}|u_1-v_1|^{2H-2}\right)$$

$$\leqslant c(H)\sum_{n\geqslant 0}(n+2)^{\alpha}nn!c_n^2\int_0^T\int_0^T du_1\,dv_2\,R(u_1,v_2)^{n-1}u_1^{-Hn}v_2^{-Hn}.$$

Using again Lemma 2 in [3], we get that the integral

$$\int_0^T\int_0^T du_1\,dv_2\,R(u_1,v_2)^{n-1}u_1^{-Hn}v_2^{-Hn}\leqslant c(H)n^{-\frac{1}{2H}}$$

and since the sequence $n!c_n^2$ behaves when $n\to\infty$ as $\frac{1}{\sqrt{n}}$ we obtain that the sum $\sum_{n\geqslant 0}(n+2)^{\alpha}\times(n+1)!B(n)$ converges if $-\alpha-\frac{1}{2}+\frac{1}{2H}>1$ and this gives $-\alpha>\frac{3}{2}-\frac{1}{2H}$. $\square$

Remark 5. Only when $H=\frac{1}{2}$ we retrieve the same order of regularity of (25) as a function of x and as a function of ω.

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